Lecture 24 : Series

So far our definition of a sum of numbers applies only to adding a finite set of numbers. We can extend this to a definition of a sum of an infinite set of numbers in much the same way as we extended our notion of the definite integral to an improper integral over an infinite interval.

Example

$$\sum_{n=1}^{\infty} \frac{1}{2^n} = \frac{1}{2} + \frac{1}{2^2} + \frac{1}{2^3} + \dots$$

We call this infinite sum a **series**

Definition Given a series $\sum_{n=1}^{\infty} a_n = a_1 + a_2 + a_3 + \dots$, we let s_n denote its n **th partial sum**

$$s_n = a_1 + a_2 + \cdots + a_n.$$

If the sequence $\{s_n\}$ is convergent and $\lim_{n\to\infty} s_n = S$, then we say that the series $\sum_{n=1}^{\infty} a_n$ is **convergent** and we let

$$\sum_{n=1}^{\infty} a_n = \lim_{n\to\infty} \sum_{i=1}^{n} a_n = \lim_{n\to\infty} s_n = S.$$

The number S is called the sum of the series. Otherwise the series is called **divergent**.

Example Find the partial sums $s_1, s_2, s_3, \dots, s_n$ of the series $\sum_{n=1}^{\infty} \frac{1}{2^n}$.
Find the sum of this series. Does the series converge?

Example Recall that $1 + 2 + 3 + \cdots + n = \frac{n(n+1)}{2}$. Does the series

$$\sum_{n=1}^{\infty} n$$

converge?

Geometric Series

The geometric series

$$\sum_{n=1}^{\infty} ar^{n-1} = a + ar + ar^2 + \cdots$$

is convergent if $|r| < 1$ and its sum is

$$\sum_{n=1}^{\infty} ar^{n-1} = \frac{a}{1-r} \qquad |r| < 1.$$

If $|r| \geq 1$, the geometric series is divergent.

Proof If $r = 1$,

$$\sum_{n=1}^{\infty} ar^{n-1} = a + a + a + a + \cdots, \qquad s_n = an, \qquad \lim_{n\to\infty} s_n = \infty$$

and the series diverges.
If $r = -1$,

$$\sum_{n=1}^{\infty} ar^{n-1} = a - a + a - a + \cdots, \quad s_n = a \text{ if n is odd and } s_n = 0 \text{ if n is even }, \quad \lim_{n\to\infty} s_n = \text{does not exist}$$

and the series diverges.

If $|r| \neq 1$, we have

$$\begin{aligned} s_n &= a + ar + ar^2 + ar^3 + \cdots + ar^{n-1} \\ rs_n &= \qquad\quad ar + ar^2 + ar^3 + \cdots + ar^{n-1} + ar^n \end{aligned}$$

Thus we get

$$s_n - rs_n = a - ar^n \quad \text{or} \quad s_n(1-r) = a(1-r^n) \quad \text{or} \quad s_n = \frac{a(1-r^n)}{(1-r)}$$

If $-1 < r < 1$, we saw in the section on sequences, that $\lim_{n\to\infty} r^n = 0$ and thus

$$\lim_{n\to\infty} s_n = \frac{a}{(1-r)}$$

giving us the desired result.

If $|r| > 1$, then $\lim_{n\to\infty} r^n$ does not exist and hence $\lim_{n\to\infty} s_n = \lim_{n\to\infty} \frac{a(1-r^n)}{(1-r)}$ does not exist. Thus the series does not converge.

Example Find the sum of the series

$$\sum_{n=1}^{\infty} \frac{(-1)^n 10}{4^{n-1}} = -10 + \frac{10}{4} - \frac{10}{16} + \ldots$$

Example Find the sum of the series

$$\sum_{n=1}^{\infty} \frac{2}{3^n} = \frac{2}{3} + \frac{2}{9} + \frac{2}{27} + \dots$$

Example Find the sum of the series

$$\sum_{n=4}^{\infty} \frac{2^{n-1}}{3^n}$$

Example Write the numbers $0.66666666\cdots = 0.\bar{6}$ and $1.521212121\cdots = 1.5\bar{2}\bar{1}$ as fractions.

Telescoping Series

These are series of the form similar to $\sum f(n) - f(n+1)$. Because of the large amount of cancellation, they are relatively easy to sum.

Example Show that the series

$$\sum_{k=1}^{\infty} \frac{1}{k^2 + 7k + 12} = \sum_{k=1}^{\infty} \frac{1}{(k+3)} - \frac{1}{(k+4)}$$

converges.

Harmonic Series

The following series, known as the harmonic series, diverges:

$$\sum_{k=1}^{\infty} \frac{1}{n}$$

We can see this if we look at a subsequence of partial sums: $\{s_{2^n}\}$.

$$s_1 = 1$$

$$s_2 = 1 + \frac{1}{2} = \frac{3}{2}$$

$$s_4 = 1 + \frac{1}{2} + \left[\frac{1}{3} + \frac{1}{4}\right] > 1 + \frac{1}{2} + \left[\frac{1}{4} + \frac{1}{4}\right] = 2$$

$$s_8 = 1 + \frac{1}{2} + \left[\frac{1}{3} + \frac{1}{4}\right] + \left[\frac{1}{5} + \frac{1}{6} + \frac{1}{7} + \frac{1}{8}\right] > s_4 + \left[\frac{1}{8} + \frac{1}{8} + \frac{1}{8} + \frac{1}{8}\right] > 2 + \frac{1}{2} = \frac{5}{2}$$

Similarly we get

$$s_{2^n} > \frac{n+2}{2}$$

and $\lim_{n\to\infty} s_n > \lim_{n\to\infty} \frac{n+2}{2} = \infty$. Hence the harmonic series diverges. (You will see an easier proof in the next section.)

Note that convergence or divergence is unaffected by adding or deleting a finite number of terms at the beginning of the series.

Example

$$\sum_{n=10}^{\infty} \frac{1}{n} \quad \text{is divergent}$$

and

$$\sum_{k=50}^{\infty} \frac{1}{2^k} \quad \text{is convergent.}$$

Divergence Test

Theorem If a series $\sum_{i=1}^{\infty} a_n$ is convergent, then $\lim_{n\to\infty} a_n = 0$.

Warning The converse is not true , we may have a series where $\lim_{n\to\infty} a_n = 0$ and the series in divergent. For example, the harmonic series.

Proof Suppose the series $\sum_{i=1}^{\infty} a_n$ is convergent with sum S. Since $a_n = s_n - s_{n-1}$ and

$$\lim_{n\to\infty} s_n = \lim_{n\to\infty} s_{n-1} = S$$

we have $\lim_{n\to\infty} a_n = \lim_{n\to\infty} s_n - \lim_{n\to\infty} s_{n-1} = S - S = 0$.

This gives us a **Test for Divergence**:

If $\lim_{n\to\infty} a_n$ does not exist or if $\lim_{n\to\infty} a_n \neq 0$, then $\sum_{i=1}^{\infty} a_n$ is divergent.

If $\lim_{n\to\infty} a_n = 0$ the test is inconclusive.

Example Test the following series for divergence with the above test:

$$\sum_{n=1}^{\infty} \frac{n^2+1}{2n^2} \qquad \sum_{n=1}^{\infty} \frac{n^2+1}{2n^3} \qquad \sum_{n=1}^{\infty} \frac{n^2+1}{2n}$$

Note that if $\lim a_n = 0$, this test is inconclusive and the series may diverge or converge.

Properties of Series

The following properties of series follow from the corresponding laws of limits:

Suppose $\sum a_n$ and $\sum b_n$ are convergent series, then the series $\sum(a_n + b_n)$, $\sum(a_n - b_n)$ and $\sum ca_n$ also converge. We have

$$\sum ca_n = c\sum a_n, \qquad \sum(a_n + b_n) = \sum a_n + \sum b_n, \qquad \sum(a_n - b_n) = \sum a_n - \sum b_n.$$

Example Sum the following series:

$$\sum_{n=0}^{\infty} \frac{3+2^n}{\pi^{n+1}}.$$

Extras

Example Sum the following series:

$$\sum_{k=1}^{\infty} \frac{1}{k^2+7k+12} - \frac{2}{3^k}.$$

Because both of these series converge we can break it into the difference of two series to sum it.

$$\begin{aligned}\sum_{k=1}^{\infty} \frac{1}{k^2+7k+12} - \frac{2}{3^k} &= \sum_{k=1}^{\infty} \frac{1}{k^2+7k+12} - \sum_{k=1}^{\infty} \frac{2}{3^k} \\ &= 1/4 - 1\end{aligned}$$

from our previous calculations.

Example For which values of x does the series $\sum_{n=1}^{\infty} \frac{x^n}{2^n}$ converge?

$\sum_{n=1}^{\infty} \frac{x^n}{2^n} = \frac{x}{2} + \frac{x^2}{2^2} + \frac{x^3}{2^3} + \ldots$. This is a geometric series with $a = \frac{x}{2} = r$. The series converges if and only if $|r| = \left|\frac{x}{2}\right| < 1$. This happens if and only if $|x| < 2$.

Example Show that the following series converges:

$$\sum_{k=1}^{\infty} \frac{1}{2^k} - \frac{1}{2^{k+1}}.$$

$$s_n = \frac{1}{2} - \frac{1}{2^2} + \frac{1}{2^2} - \frac{1}{2^3} + \cdots + \frac{1}{2^n} - \frac{1}{2^{n+1}} = \frac{1}{2} - \frac{1}{2^{n+1}}$$

$$\sum_{k=1}^{\infty} \frac{1}{2^k} - \frac{1}{2^{k+1}} = \lim_{n\to\infty} s_n = \lim_{n\to\infty} \frac{1}{2} - \frac{1}{2^{n+1}} = \frac{1}{2}$$

Therefore the series converges to $\frac{1}{2}$.

Puzzle for the Friday Night Calc Party: Ant on a rubber band. An ant starts at one end of a one meter rubber band, placed conveniently at $x = 0$ on the x axis. Initially the other end of the rubber band is at $x = 1$. Each second the ant walks 1 cm. At the end of each second Carlito, who likes teasing ants, stretches the rubber band by one meter. (Note the point at which the ant is at moves when the band is stretched.). Will the ant ever reach the end of the rubber band?

Hint calculate the proportion of the distance covered by the ant after 1 second, 2 seconds 3 seconds, ..., n seconds, and derive your answer from the sum of the series.